Green's identities and integration by parts

Katharine Long

Let $\Omega \in \mathbb{R}^d$ be a region with boundary Γ, and let f and $\mathbf{G}$ be a scalar and vector field, respectively.

1 Ingredients

Green's identities are based on two building blocks: the Divergence Theorem and a product rule for the divergence. You'll recall the Divergence Theorem relates a volume integral to a surface integral,

$$\int_\Omega \nabla \cdot \mathbf{G}\, d\Omega = \int_\Gamma \mathbf{G} \cdot d\mathbf{\Gamma}.$$

The product rule for the divergence is

$$\nabla \cdot (f\mathbf{G}) = \mathbf{G} \cdot \nabla f + f \nabla \cdot \mathbf{G},$$

which you can easily check by direct calculation (easiest in Cartesian coordinates, and 2D is sufficient). With these ingredients, we can construct integration by parts for volume integrals involving the divergence.

2 Integration by parts

2.1 Review of IBP in one variable

In one-variable calculus, integration by parts combines the product rule and the fundamental theorem of calculus. Since

$$\frac{d}{dx}(fg) = f'g + fg',$$

we can calculate

$$\begin{aligned}\int_a^b f'g + fg'\, dx &= \int_a^b \frac{d}{dx}(fg)\, dx \\ &= fg|_a^b,\end{aligned}$$

and then after a little rearrangement find

$$\int_a^b fg'\, dx = fg|_a^b - \int_a^b f'g\, dx.$$

2.2 IBP with the divergence

Now apply the same idea with surface and volume integrals. Compute

$$\begin{aligned}\int_\Omega f\nabla\cdot\mathbf{G}+\mathbf{G}\cdot\nabla f\,d\Omega &= \int_\Omega \nabla\cdot(f\mathbf{G})\,d\Omega\\ &= \int_\Gamma f\mathbf{G}\cdot d\mathbf{\Gamma},\end{aligned}$$

which upon rearrangement becomes

$$\int_\Omega f\nabla\cdot\mathbf{G}\,d\Omega = \int_\Gamma f\mathbf{G}\cdot d\mathbf{\Gamma} - \int_\Omega \mathbf{G}\cdot\nabla f\,d\Omega. \tag{1}$$

Notice that the structure of this calculation is identical to that of IBP in one variable.

3 Green's identities

Green's identities are obtained by letting $\mathbf{G}$ be the gradient of a scalar function g. Set $\mathbf{G}=\nabla g$, and substitute into equation 1. Then we find **Green's first identity**,

$$\int_\Omega f\nabla^2 g\,d\Omega = \int_\Gamma f\nabla g\cdot d\mathbf{\Gamma} - \int_\Omega \nabla f\cdot\nabla g\,d\Omega. \tag{2}$$

Next, use Green's first identity to integrate $f\nabla^2 g - g\nabla^2 f$,

$$\int_\Omega f\nabla^2 g - g\nabla^2 f\,d\Omega = \int_\Gamma (f\nabla g - g\nabla f)\cdot d\mathbf{\Gamma}, \tag{3}$$

where the two integrals involving $\nabla f\cdot\nabla g$ have canceled. This is – you guessed it – **Green's second identity**.