

Our first PDE: the heat equation

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Suppose $u(\mathbf{r}, t)$ is the temperature at position $\mathbf{r}$ and time t . In a material that conducts heat, heat will flow from hot regions to cold regions; the vector $\mathbf{q}$ describing heat flow is therefore

$$\mathbf{q} = -\kappa \nabla u \quad (1)$$

where κ is a material-dependent constant called the **thermal conductivity**. In math class, we set material constants like this to 1. Equation 1 is called Fourier's Law of Heat Conduction. The density of thermal energy is proportional to temperature, $c_v u$; again we set the **volumetric heat capacity** c_v to 1. The relation between heat flow and rate of change of energy density is expressed by the **continuity equation**

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{q} = 0.$$

With Fourier's Law for $\mathbf{q}$, we find

$$\frac{\partial u}{\partial t} - \nabla \cdot \nabla u = 0$$

or

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

This is called the **heat equation**. To keep it simple start in one spatial dimension, so our equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \quad (2)$$

We'll often use the shorthand notation u_t for $\frac{\partial u}{\partial t}$ and u_{xx} for $\frac{\partial^2 u}{\partial x^2}$, in which case we can write the heat equation as $u_t = u_{xx}$.

The heat equation usually comes with **boundary conditions** such as

$$u(0, t) = u(L, t) = 0 \quad \forall t \quad (3)$$

and **initial conditions**

$$u(x, 0) = U_0(x). \quad (4)$$

Equation 2 involves partial derivatives of the unknown $u(x, t)$ with respect to two independent variables, x and t , so it's a **partial differential equation** (PDE). Notice also that the equation is linear and homogeneous, so we can use superposition: any linear combination of solutions is a solution.

1 Thinking before calculating

Suppose we have the heat equation with Dirichlet-Dirichlet BCs as in equation 3, and an initial temperature profile $u(x, 0) = U_0(x)$. The solution at some small time Δt after $t = 0$ will be roughly

$$\begin{aligned} u(x, \Delta t) &\approx U_0(x) + \left. \frac{\partial u}{\partial t} \right|_{t=0} \Delta t \\ &\approx U_0(x) + \left. \frac{\partial^2 u}{\partial x^2} \right|_{t=0} \Delta t \\ &\approx U_0(x) + U_0''(x) \Delta t. \end{aligned}$$

Then:

- Where the profile is concave down ($U_0''(x) < 0$) the temperature will decrease with time.
- Where the profile is concave up ($U_0''(x) > 0$) the temperature will increase with time.

Any bumps in the initial profile will smooth out (see figure 1). The greater the second derivative, the faster the change, so sharper bumps smooth out faster.

Now that we have some idea what to expect, let's solve the problem.

2 Solution procedure for the heat equation

When encountering a new kind of math problem, the best strategy is to find a way to turn it into a familiar problem that you know how to solve. That's what exactly we'll do here. The approach will be reminiscent of how we solved systems of linear ODEs $\mathbf{x}' = A\mathbf{x}$,

2.1 Separation of variables

Similar to how we tried $\mathbf{v}f(t)$ for the ODE system $\mathbf{x}' = A\mathbf{x}$, let's try a **product solution** $u(x, t) = S(x)T(t)$. Here S (for "space") depends only on x , while T (for "time") depends only on t . Then

$$\frac{\partial^2 u}{\partial x^2} = S''(x)T(t); \quad \frac{\partial u}{\partial t} = S(x)T'(t).$$

Plug these partials into the heat equation,

$$\begin{aligned} u_t &= u_{xx} \\ S(x)T'(t) &= S''(x)T(t) \end{aligned}$$

divide by ST to put all time dependence on the LHS and all space dependence on the RHS,

$$\frac{T'(t)}{T(t)} = \frac{S''(x)}{S(x)}.$$

Since the LHS is independent of x and the RHS is independent of t , both must be a constant, which we'll call $-\lambda$. Thus we find

$$\frac{T'(t)}{T(t)} = -\lambda; \quad \frac{S''(x)}{S(x)} = -\lambda.$$

The spatial problem $-S'' = \lambda S$ together with is an eigenvalue problem. Recall that we showed that when we have homogeneous boundary conditions, the eigenvalues of $-S''$ must be nonnegative, so let's set $\lambda = k^2$. Then the eigenvalue problem for S and k^2 is the harmonic oscillator problem

$$S''(x) + k^2 S(x) = 0$$

together with boundary conditions. We know how to do this problem! The equation for $T(t)$ is

$$T'(t) = -k^2 T(t),$$

which we also know how to do: the solution is

$$T(t) = e^{-k^2 t}$$

times any constant.

2.2 Solving the spatial eigenvalue problem

2.2.1 Finding the boundary condition

We're given boundary conditions for $u(x, t)$. We need boundary conditions for $S(x)$. Suppose the BCs are as in equation 3,

$$u(0, t) = u(L, t) = 0 \quad \forall t.$$

With the product solution this takes the form

$$S(0) T(t) = S(L) T(t) = 0 \quad \forall t.$$

Clearly setting $T(t) = 0 \quad \forall t$ would work, but that solution would be trivial. Therefore, set

$$S(0) = S(L) = 0. \tag{5}$$

2.2.2 Solving the eigenvalue problem

You know how to solve $S'' + k^2 S = 0$ with the BCs $S(0) = S(L) = 0$: the eigenfunctions are

$$v_n(x) = \sin k_n x$$

with eigenvalues

$$k_n^2 = \left(\frac{n\pi}{L}\right)^2$$

for $n \in \mathbb{N}_1$. Recall that the operator $-\frac{d^2}{dx^2}$ on a subspace obeying these BCs is Hermitian, so the eigenfunctions are orthogonal.

2.3 Forming the general solution

With the eigenvalues k_n^2 known, the time part $T_n(t)$ of the product solution for the n -th eigenvalue is

$$T_n(t) = e^{-k_n^2 t}.$$

The space part $S_n(x)$ is the eigenfunction $v_n(x)$, so the product solution is

$$S_n(x) T_n(t) = v_n(x) e^{-k_n^2 t}.$$

The general solution is a superposition of product solutions,

$$u_g(x, t) = \sum_{n=1}^{\infty} c_n v_n(x) e^{-k_n^2 t}.$$

2.4 Determining the expansion coefficients

We have initial conditions $u(x, 0) = U_0(x)$, so evaluate $u_g(x, t)$ at $t = 0$ and set equal to the initial value $U_0(x)$,

$$\sum_{n=1}^{\infty} c_n v_n(x) = U_0(x).$$

The eigenfunctions are orthogonal, so we can find c_m by taking inner products of both sides with v_m ,

$$\sum_{n=1}^{\infty} c_n \langle v_m, v_n \rangle = \langle v_m, U_0 \rangle$$

and then

$$c_n = \frac{\langle v_n, U_0 \rangle}{\langle v_n, v_n \rangle}.$$

All done!

Notice that although different boundary conditions will give different eigenvalues and eigenfunctions than those in this simple example, the procedure will be the same.

3 Example

Suppose we have Dirichlet BCs at $x = 0$ and $x = 1$, and initial condition $u(x, 0) = U_0(x)$ with

$$U_0(x) = \begin{cases} 0 & x < \frac{1}{4} \\ 1 & \frac{1}{4} < x < \frac{3}{4} \\ 0 & x > \frac{3}{4} \end{cases}.$$

This function is plotted in figure 2. The eigenfunctions are $v_n(x) = \sin k_n x$ with $k_n = n\pi$, so the solution is

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-n^2 \pi^2 t}.$$

The coefficients in the solution are obtained by computing

$$c_n = \frac{\langle v_n, U_0 \rangle}{\langle v_n, v_n \rangle}.$$

The denominator is

$$\int_0^1 \sin^2(n\pi x) dx = \int_0^1 \frac{1}{2} (1 - \cos(2n\pi x)) dx = \frac{1}{2}$$

and the numerator is

$$\int_{\frac{1}{4}}^{\frac{3}{4}} \sin(n\pi x) dx = -\frac{1}{n\pi} \cos(n\pi x) \Big|_{\frac{1}{4}}^{\frac{3}{4}} = \frac{1}{n\pi} \left(\cos \frac{n\pi}{4} - \cos \frac{3n\pi}{4} \right).$$

For reference, the first few nonzero coefficients are

n	c_n
1	$\frac{2\sqrt{2}}{\pi}$
3	$-\frac{2\sqrt{2}}{3\pi}$
5	$-\frac{2\sqrt{2}}{5\pi}$
7	$\frac{2\sqrt{2}}{7\pi}$
9	$\frac{2\sqrt{2}}{9\pi}$

so the solution is

$$u(x, t) = \frac{2\sqrt{2}}{\pi} \left[e^{-\pi^2 t} \sin \pi x - \frac{1}{3} e^{-9\pi^2 t} \sin 3\pi x - \frac{1}{5} e^{-25\pi^2 t} \sin 5\pi x + \frac{1}{7} e^{-49\pi^2 t} \sin 7\pi x + \dots \right].$$

Notice that at $t = 0.1$, the ratio of the second term to the first is

$$-\frac{1}{3} e^{-0.8\pi^2} \approx 1.24 \times 10^{-4}$$

and the ratio of the third term to the second is

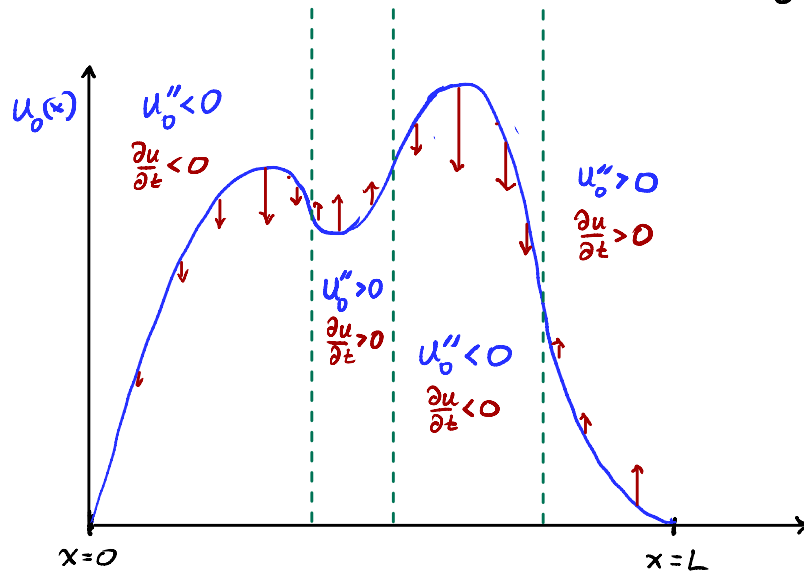
$$\frac{3}{5} e^{-1.7\pi^2} \approx 3.1 \times 10^{-8}.$$

Even for small t , these terms decrease *very* fast with n . The high-frequency terms die quickly; any bumps or sharp features require high-frequency terms, so bumps and sharp jumps get smoothed out very quickly, just as we deduced before solving the equation.

4 Summary

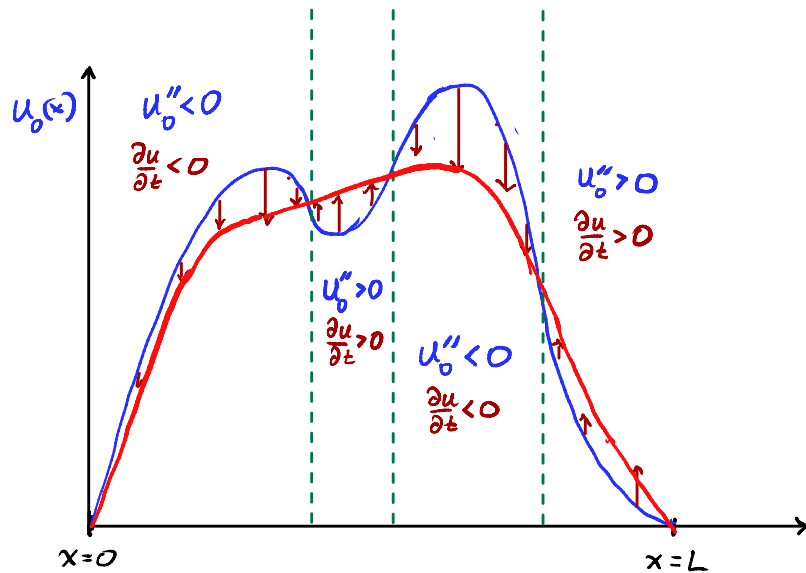
We used a solution procedure for $u_t = u_{xx}$ that resulted in an expansion in exponentials in time times eigenfunctions in space, just like we solved $\mathbf{x}' = A\mathbf{x}$ by finding a linear combination of eigenvectors times exponentials in time. Because the operator $-\frac{d^2}{dx^2}$ is Hermitian, the eigenfunctions are orthogonal and therefore we can find coefficients using inner products.

Blue shows initial value $u(x,0)=u_0(x)$.



Dark red shows $\Delta u = \frac{\partial u}{\partial t} \Delta t = u_0''(x) \Delta t$.

Red shows $u(x,0+\Delta t)$ for a small Δt



"Bumps" in $u(x,t)$ get smoothed out.

Figure 1: Before solving the heat equation, we can deduce that any bumpy features in the solution will smooth out as time increases.

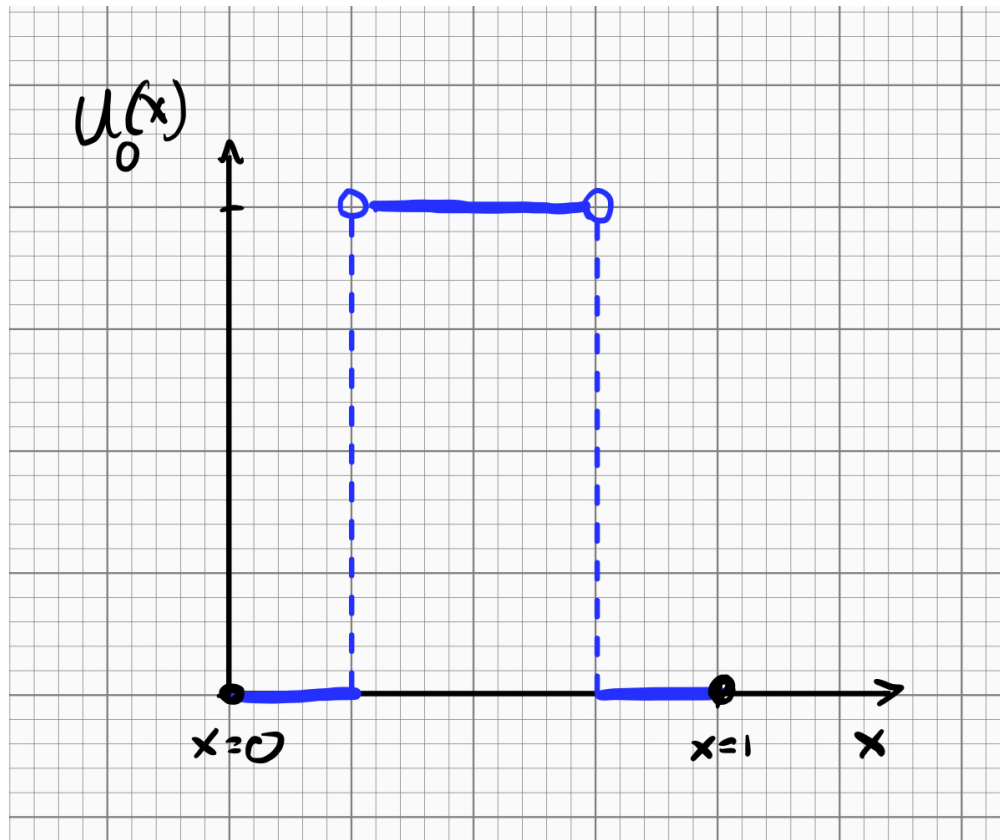


Figure 2: The initial profile $U_0(x)$