Eigenvalues of the Helmholtz equation

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These notes are intended to establish the following fact: the negative Laplacian operator $-\nabla^2$ has non-negative eigenvalues.

1 The Helmholtz equation

The scalar Helmholtz equation

$$\nabla^2 u + \lambda u = 0$$

appears naturally in many physics problems, for example, if one looks for product solutions $u(\mathbf{r}, t) = S(\mathbf{r}) T(t)$ to the heat or wave equation, the spatial part $S(\mathbf{r})$ must be a solution to the Helmholtz equation

$$\nabla^2 S + \lambda S = 0.$$

We'll There is also a generalized Helmholtz equation

$$\nabla \cdot (\kappa(\mathbf{r}) \nabla u) + \lambda \sigma(\mathbf{r}) u = 0,$$

in which the coefficient functions κ and σ are strictly positive (except perhaps at the boundary). This will appear, for example, in problems where material properties (*e.g.* thermal conductivity) vary with space. Although the generalized Helmholtz equation will be more difficult to solve than the ordinary Helmholtz equation, the conclusions we'll draw here about eigenvalues still apply to it.

1.1 Boundary conditions

In an eigenvalue problem the Helmholtz equation will always be accompanied by homogeneous boundary conditions. The two most common types of boundary conditions are:

- Specifying the function u on the boundary (called **Dirichlet conditions**). If homogeneous Dirichlet conditions are imposed on a boundary subset $\Gamma_D \subseteq \Gamma$, we write them as
$$u = 0 \text{ on } \Gamma_D.$$
- Specifying the normal component of ∇u on the boundary (called **Neumann conditions**). If Neumann conditions are imposed on a boundary subset $\Gamma_N \subseteq \Gamma$, we write them as
$$\hat{\boldsymbol{n}} \cdot \nabla u = 0 \text{ on } \Gamma_N.$$

Less common is to specify $\hat{\boldsymbol{n}} \cdot \nabla u + \beta u = 0$ on all or part of the boundary (called **Robin conditions**). The function $\beta(\mathbf{r})$ must be strictly positive on the boundary subsets on which this condition is to be used. Finally, there are **periodic boundary conditions** in which the solution is forced to be periodic in one of more of the spatial coordinate.

Here I'll consider only Dirichlet and Neumann conditions, so that Γ_D and Γ_N together make up the whole boundary, which we can write as $\Gamma_D \cup \Gamma_N = \Gamma$. It can happen that there are no Neumann conditions anywhere, in which case $\Gamma_N = \emptyset$, or that there are no Dirichlet conditions anywhere, in which case $\Gamma_D = \emptyset$. The conclusion that all eigenvalues are non-negative is valid for periodic and Robin boundary conditions, but I'll leave that for you to prove.

2 Intuition from 1D

In one dimension, the Helmholtz reduces to the harmonic oscillator equation

$$\frac{d^2v}{dx^2} + \lambda v = 0,$$

which we'll write in the equivalent form

$$-\frac{d^2v}{dx^2} = \lambda v \tag{1}$$

which has the structure $Lv = \lambda v$, with L being the differential operator $-\frac{d^2}{dx^2}$. Suppose Dirichlet or Neumann boundary conditions are imposed at $x = a$ and $x = b$, and let $v(x)$ be an eigenfunction. Now consider some point $x \in (a, b)$ at which $v(x) > 0$, and examine the implications for the curvature of v.

- If $\lambda > 0$, then from equation 1 we see $v''(x)$ is negative and $v(x)$ is concave down, *i.e.*, towards the x axis.
- If $\lambda < 0$, then from equation 1 we see $v''(x)$ is positive and $v(x)$ is concave up, *i.e.*, away from the x axis.

Choose another point at which $v(x) < 0$, and you'll find the same result: if $\lambda > 0$ then $v(x)$ must be concave towards the x axis, while if $\lambda < 0$ then $v(x)$ must be concave away from the axis.

But if $v(x)$ is everywhere concave away from the x-axis, it can't possibly satisfy any combination of Dirichlet or Neumann boundary conditions; this is easiest to see with Dirichlet conditions at both boundaries. Since $\lambda < 0$ implies $v(x)$ is everywhere concave away from the axis, it is therefore impossible for λ to be negative.

3 A more general proof

The argument in section 2 is valid for Cartesian coordinates in one dimension. Now let's develop a proof valid in any coordinate system and any number of dimensions. You might want to refer to my notes on Green's identities which will help with one step in the derivation.

Let $\langle \cdot, \cdot \rangle$ be the inner product

$$\langle \overline{u}, v \rangle = \int_\Omega \overline{u} v \, d\Omega.$$

Suppose v is an eigenfunction, and form the inner product

$$\left\langle v, \nabla^2 v + \lambda v \right\rangle$$

which is clearly zero since v is a solution to the Helmholtz equation. Therefore,

$$\langle v, \lambda v \rangle = \left\langle v, -\nabla^2 v \right\rangle,$$

or

$$\lambda = \frac{\left\langle v, -\nabla^2 v \right\rangle}{\langle v, v \rangle}.$$

Notice that this is a Rayleigh quotient! Write out the inner products with integrals: in the denominator,

$$\langle v, v \rangle = \int_\Omega |v(\mathbf{r})|^2 \, d\Omega$$

and in the numerator,

$$\left\langle v, -\nabla^2 v \right\rangle = -\int_\Omega \overline{v} \nabla^2 v \, d\Omega.$$

Now apply Green's first identity to obtain

$$\begin{aligned} \left\langle v, -\nabla^2 v \right\rangle &= \int_\Omega \overline{\nabla v} \cdot \nabla v \, d\Omega - \int_\Gamma \overline{v} \hat{\boldsymbol{n}} \cdot \nabla v \, d\Gamma \\ &= \int_\Omega |\nabla v|^2 \, d\Omega - \int_\Gamma \overline{v} \hat{\boldsymbol{n}} \cdot \nabla v \, d\Gamma. \end{aligned}$$

With Dirichlet or Neumann boundary conditions, either $v = 0$ or $\hat{\boldsymbol{n}} \cdot \nabla v = 0$ on every point on Γ, so we find

$$\lambda = \frac{\left\langle v, -\nabla^2 v \right\rangle}{\langle v, v \rangle} = \frac{\int_\Omega |\nabla v|^2 \, d\Omega}{\int_\Omega |v|^2 \, d\Omega}$$

which cannot be negative.

4 For you to try

We made two simplifying assumptions:

- Boundary conditions are Dirichlet and/or Neumann, not periodic or Robin.
- We used the standard Helmholtz equation $\nabla^2 u + \lambda u = 0$, not the generalized equation

  $$\nabla \cdot (\kappa \nabla u) + \lambda \sigma u = 0$$

  in which $\kappa(\mathbf{r}) > 0$ and $\sigma(\mathbf{r}) > 0$ are material properties that might vary in space.

See if you can extend the argument in section 3 to show that all eigenvalues of the generalized Helmholtz equation are non-negative, given any combination of Dirichlet, Neumann, Robin, or periodic boundary conditions.